Anton Korinek

Mathematical Review

Review of Financial Mathematics

1) Compounding

Example 1: What is the future value (FV) of $100 if we invest it at an interest rate of r = 5% for one year?

Answer: FV = $100 + ($100 * 5%) = $100 * (1 + 5%) = $100 * 1.05 =
(principal: $100; interest: $100 * 5%)

Example 2: What is the final value of a $1000 investment at an interest rate of 7% after 3 years?

Answer: FV = [($1000 * 1.07) * 1.07] * 1.07 = $1000 * 1.07^3 =

> *General notation:*
>
> PV ......Present value (or starting value, current value)
> FV ......Future value (or final value, payoff)
> r ....Interest rate (or rate of return, discount rate)
> n..........Number of time periods (usually years)
>
> *General formula:* $FV = PV * (1 + r)^n$

Example 3: What is the final value of a $89 investment that yields a 7% return in the first year and a 5% return in the second year?

Answer: FV = $89 * 1.07 * 1.05 =

Example 4: Assume the current price of a stock is $400. The stock price rises by 8% annually for three years, and then it declines by 25%. What is the final value of the stock?

Answer: FV = $400 * $(1 + .08)^3$ * (1 – .25) =

2) Sub-Annual Compounding

Example 5: How much do we have on a savings book with $100 after one year, if the annual interest rate is r = 6% and interest is compounded (a) annually (b) quarterly?

Answer: "Annual compounding" means interest is paid at the end of the year:
FV = $100 * 1.06 =

"Quarterly compounding" means ¼ of the interest rate is paid every quarter:
FV = \$100 * (1 + 6%/4) * (1 + 6%/4) * (1 + 6%/4) * (1 + 6%/4) =
= \$100 * 1.015^4 =

> *General notation:*
>
> r ....Annual interest rate
> m ....Number of compounding periods per year
> n..........Total number of compounding periods
>
> *General formula:* $FV = PV * (1 + r/m)^n$

Example 6: Assume you have a \$1000 balance on a credit card with no monthly payment and a 36% interest rate that is compounded monthly. What is your debt after two years?

Answer: $FV = \$1000 * (1 + 36\%/12)^{2*12} = \$1000 * 1.03^{24} =$

3) Continuous Compounding

The more frequently we compound interest, the faster the final value of an investment grows. The table below indicates the final value for a \$100 investment that earns an annual interest rate of r = 12% for different compounding frequencies:

Frequency	**Annual**	**Quarterly**	**Monthly**	**Daily**	**…**	**Continuously**
Compounding periods per year	1	4	12	360	…	∞
FV after 1 year	112.00	112.55	112.68	112.75	…	112.75
FV after 10 years	310.58	326.20	330.04	331.95	…	332.01

In the limit, as the number of compounding periods grows towards infinity, the formula for the final value after t years (i.e. after t*m compounding periods) becomes:

$$FV = \lim PV * (1 + r/m)^{t*m} = PV * e^{r*t}$$

Example 7: How much do we have on a savings book with \$100 after one year, if the annual interest rate is r = 6% and interest is compounded continuously?

Answer: $FV = \$100 * e^{.06} =$

4) Discounting

> *General formula:* $PV = FV / (1 + r)^n$

Example 8: What is the present value of a final payoff of \$105 in one year from now, given an interest rate of 5%?

Answer: PV = \$105 / (1 + .05) =

Example 9: How much money do you have to put into a savings account now if you want to have $100,000 when you retire in 30 years, given an interest rate of 5%?

Answer: PV = $\$100{,}000 / 1.05^{30} =$

5) Rates of Return

Example 10: You bought a stock for $400 and sold it for $440 a year later. What was the return?

Answer: $r = \$440/\$400 - 1 =$

> *General formula:* $1 + r = \sqrt[n]{FV / PV}$

Example 11: You bought a stock for $400 and sold it for $480 two years later. What was the total rate of return? What was the annual rate of return?

Answer: Total rate of return $= \$480 / \$400 - 1 =$
Annual rate of return $= \sqrt[2]{\$480 / \$400} - 1 =$

Example 12: You bought a stock for $400 and sold it for $800 ten years later. What was the total rate of return? What was the annual rate of return?

Answer: Total rate of return $= \$800 / \$400 - 1 =$
Annual rate of return $= \sqrt[10]{\$800 / \$400} - 1 =$

6) Geometric Series

Example 13: You put $500 into a savings book at year-end for each of four years. What will be the final value, given an interest rate of 5%?

Answer: $FV = \underbrace{\$500 * 1.05^{3}}_{\text{year 1}} + \underbrace{\$500 * 1.05^{2}}_{\text{year 2}} + \underbrace{\$500 * 1.05}_{\text{year 3}} + \underbrace{\$500}_{\text{year 4}} =$

> *General notation:*
>
> FV ......Final value
> P ....Amount of yearly payment (at year-end)
> r ....Interest rate
> n..........Number of payments
>
> *General formula:* $FV = P * \frac{(1+r)^n - 1}{r}$

Answer 2: $FV = \$500 * \frac{1.05^4 - 1}{.05} =$

Example (ctd.): If you save the $500 at the beginning of each year instead of at year-end, what is the final value?

Answer: FV = $500 * 1.05 ^{4} + $500 * 1.05 ^{3} + $500 * 1.05 ^{2} + $500 * 1.05 =
(year 1, year 2, year 3, year 4)

$$= \$500 * 1.05 * \frac{1.05^4 - 1}{.05} =$$

Example 14: You win a yearly payment of $1000 (at year-end) from a lottery for the next 20 years. What is the present value of this, given an interest rate of 4%?

Answer: $\text{FV} = \$1000 * \frac{1.04^{20} - 1}{.04} =$

PV = FV / 1.04^{20} =

Review of Statistics

7) Random Variables

Definition: A random variable can take different values depending on some (random) state of nature. We describe a random variable X with a set of n possible outcomes $\{x_1, x_2, \ldots x_n\}$ that occur with probabilities $\{p_1, p_2, \ldots p_n\}$ each.

Example 15: D = outcome of rolling a die:
n = 6 possible outcomes: $d_1 = 1$, $d_2 = 2$, … $d_6 = 6$ with probability $p_i = 1/6$ each

Example 16: X = outcome of a coin toss:
n = 2 possible outcomes: $x_1 = 0$ if head or $x_2 = 1$ if tail with probability $p_i = ½$

Example 17: S = price of a stock:
n = 4 possible outcomes: $s_i \in \{50, 80, 100, 130\}$ with probabilities {.1, .1, .5, .3}

8) Expected Value

Definition: The expected value E[X] captures the "average" realization of a random variable X.

General formula: $E[X] = p_1*x_1 + p_2*x_2 + \ldots + p_n*x_n = \sum_i p_i x_i$

Example 18: Coin toss: E[X] = ½*0 + ½*1 =

Example 19: Rolling a die: E[X] = 1/6*1 + 1/6*2 + 1/6*3 + 1/6*4 + 1/6*5 + 1/6*6 =

Example 20: Stock price: E[P] = .1*50 + .1*80 + .5*100 + .3*130 =

A few useful results (for random variables X and Y and constants a and b):

- E[a] = a
- E[a*X + b*Y] = a*E[X] + b*E[Y]
- E[X – E[X]] = 0

9) Variance

Definition: The variance Var(X) or σ^2_X measures how much a random variable is dispersed around its expected value. More specifically, it captures the average squared deviation of a random variable from its expected value.

General formula:

$$\text{Var}(X) = E[(X - E[X])^2] =$$
$$= p_1*(x_1 - E[X])^2 + p_2*(x_2 - E[X])^2 + \ldots + p_n*(x_n - E[X])^2 =$$
$$= \sum_i p_i (x_i - E[X])^2$$

Example 21: Stock price P:

$$\text{Var}(P) = .1*(50 - 102)^2 + .1*(80 - 102)^2 + .5*(100 - 102)^2 + .3*(130 - 102)^2 =$$

A few *useful results* (for a random variable X and a constant a):

- Var(a) = 0
- $\text{Var}(a*X) = a^2*\text{Var}(X)$
- Var(X + a) = Var(X)

Definition: The standard deviation σ_X of a random variable X is the square root of its variance σ^2_X.

Example 22: $\sigma_P = \sqrt{\sigma^2_P} =$

10) Covariance and Correlation

Definition: The covariance Cov(X, Y) or $\sigma_{X,Y}$ measures the extent to which two random variables move together (co-move). It is positive if the random variables usually tend to move in the same direction, and negative if they tend to move in opposite directions.

General formula:

$$\text{Cov}(X,Y) = E[(X - E[X])*(Y - E[Y])] =$$
$$= p_1*(x_1 - E[X])^* (y_1 - E[Y]) + \ldots + p_n*(x_n - E[X])* (y_n - E[Y]) =$$
$$= \sum_i p_i (x_i - E[X])(y_i - E[Y])$$

A few *useful results* (for random variables X and Y and positive constants a and b):

- Cov(X, a) = 0
- Cov(X, X) = Var(X)
- Cov(X, Y) = Cov(Y, X)
- Cov(X + a, Y + b) = Cov(X, Y)
- Cov(a*X, b*Y) = a*b*Cov(X, Y)
- Var(X + Y) = Var(X) + Var(Y) + 2*Cov(X, Y)

Definition: The correlation Corr(X,Y) or $\rho_{X,Y}$ normalizes the covariance by dividing it through the standard deviations σ_X and σ_Y so that it always lies between –1 and +1.
If Corr(X,Y) = –1, then the two random variables X and Y always move in opposite directions; if Corr(X,Y) = 0, then they move independently from each other, i.e. they are *uncorrelated*, and if Corr(X,Y) = 1, then they always move in the same direction, i.e. they are *perfectly correlated.*

General formula: $\text{Corr}(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$

A few *useful results* (for random variables X and Y and positive constants a and b):

- Corr(X, a) = 0
- Corr(X, X) = 1
- Corr(X, Y) = Corr(Y, X)
- Corr(X + a, Y + b) = Corr(a*X, b*Y) = Corr(X, Y)

Example 23: Assume there are three stocks X, Y, Z and three possible states of the world {normal, boom and recession} that occur with equal probability $p_i = 1/3$. The prices of the three stocks are:

State	x_i	y_i	z_i	x_i – E[X]	y_i – E[Y]	z_i – E[Z]
normal	100	90	50			
boom	110	100	90			
recession	90	110	10			
E[stock]						
Var(stock)						
σ_{stock}						

1) What are the covariances and correlations of (X,Y) and of (X,Z)?
2) What is the variance Var(X + Y)?

Answer: Cov(X,Y) = 1/3*0*(–10) + 1/3*10*0 + 1/3*(–10)*10 =

Cov(X,Z) = 1/3*0*0 + 1/3*10*40 + 1/3*(–10)* (–40) =

$\text{Corr}(X,Y) = \dfrac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} =$

$\text{Corr}(X,Z) = \dfrac{\text{Cov}(X,Z)}{\sigma_X \sigma_Z} =$

Var(X + Y) = Var(X) + Var(Y) + 2*Cov(X, Y) =

11) Independent Random Variables

Definition: Two random variables are independent if the outcome of one variable contains no information about the outcome of the other variable.

Examples: rolling a fair die multiple times
tossing several fair coins

Note: The covariance of independent random variables is always zero.